ODE TUT 2

Revision:

uniquness and existence thm:

let f and $\frac{\partial f}{\partial y}$ be cts in some rectangle $\alpha<t<\beta$, $\gamma<y<\delta$, containing (t_0, y_0), then in some interval $t_0-h<t<t_0+h$ in $\alpha<t<\beta$, $\exists$ unique solution $y=\phi(t)$ s.t $\begin{cases} y'=f(t,y) \\ y(t_0)=y_0 \end{cases}$

(If the equation is in form of $y'+Py=g$, it will have a unique solution if p and g are cts).

Def: A equation in form of $y'=f(y)$ is called autonomous,

Def: The constant function satisfies the autonomus ODE is called equilibrium solution and the zeros of $f(y)$ are critical point.

Def: let $M(x,y)dx + N(x,y)dy = 0$, —①

If $\exists\, \psi(x,y)$ s.t. $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$, then

① is called an exact ODE.

Thm: If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are cts in the rectangular region $\alpha < x < \beta$, $\gamma < y < \delta$, then

① is exact iff $M_y = N_x$.

We can convert a ODE that is not exact into an exact ODE by multiplying a integrating factor.

If μ is an integrating factor, then

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu \quad \text{or} \quad \frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

try to solve it,

If $\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$; then

$\mu M dx + \mu N dy = 0$ is exact

$(uM)_y = uM_y$

$= N\frac{du}{dx} + N_x u = (Nu)_x$

Second order ODE

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

A 2nd order ODE is homogeneous if $G(t)=0$, that is $P(t)y''+Q(t)y'+R(t)y=0$

If P, Q, R are constant, say

$$ay''+by'+cy=0 \quad , a\neq 0$$

step 1: let the characteristic equation be $a\lambda^2+b\lambda+c=0$,

find the roots of λ.

step 2, Case 1 $\lambda = r_1$ or r_2, $r_1 \neq r_2$.

both are real,

the solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, c_1, c_2 are constant.

Case 2 $\lambda = r \in \mathbb{R}$ double roots,

$$y = (c_1 + c_2 t) e^{rt}$$

Case 3 $\lambda = v \pm i\mu$

$$y = e^{vt}(c_1 \cos \mu t + c_2 \sin \mu t)$$

Problem

① Determine if they are exact, If exact, find a solution,

$\boxed{1a}$: $(x^4 + 4y) + (4x - 3y^8) y' = 0$

Ans: $\frac{\partial}{\partial y}(x^4 + 4y) = 4$

$\frac{\partial}{\partial x}(4x - 3y^8) = 4$, it is exact,

$\exists\ \psi$ s.t., $\frac{\partial \psi}{\partial x} = x^4 + 4y$

$$\psi = \frac{x^5}{5} + 4xy + h(y)$$

and $\frac{\partial \psi}{\partial y} = 4x + h'(y) = 4x - 3y^8$

$$h = -\frac{3}{9} y^9 + C$$

$\therefore$ $\psi(x,y) = C$ is a solution

$$\frac{x^5}{5} + 4xy - \frac{1}{3} y^9 = C$$

$\boxed{1b}$: $(2xy^2+2y)+(2x^2y+2x)y'=0$

Ans: $\frac{\partial}{\partial y}(2xy^2+2y)=4xy+2$

$\frac{\partial}{\partial x}(2x^2y+2x)=4xy+2$

It is exact

So $\frac{\partial \psi}{\partial x}=2xy^2+2y$

$\psi=x^2y^2+2xy+h(y)$

$\frac{\partial \psi}{\partial y}=2x^2y+2x+h'(y)=2x^2y+2x$

$h=C$.

$x^2y^2+2xy=C$.

2 use integrating factor to solve,

$\boxed{1a}$ $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0$

Ans: $\frac{\partial}{\partial y}(3x^2y+2xy+y^3)=3x^2+2x+3y^2$

$\frac{\partial}{\partial x}(x^2+y^2)=2x$, It is not exact,

$\frac{du}{dx}=\frac{3x^2+2x+3y^2-2x}{x^2+y^2}u=3u$

$\therefore u=e^{+3x}$ (Constant is not needed).

Multiple e^{3x},

$$e^{3x}(3x^2y+2xy+y^3)dx+e^{3x}(x^2+y^2)dy=0$$

$\exists \psi$, s.t. $\dfrac{\partial\psi}{\partial x}=e^{3x}(3x^2y+2xy+y^3)$

$$\psi=3y\int e^{3x}x^2dx+2y\int e^{3x}x\,dx+y^3\int e^{3x}dx+h(y)$$

$$=y\int x^2de^{3x}+2y\int e^{3x}x\,dx+y^3\frac{e^{3x}}{3}+h(y)$$

$$=y\left(x^2e^{3x}-2\int e^{3x}x\,dx\right)+2y\int e^{3x}x\,dx+y^3\frac{e^{3x}}{3}+h(y)$$

$$=yx^2e^{3x}+y^3\frac{e^{3x}}{3}+h(y).$$

and $\dfrac{\partial\psi}{\partial y}=x^2e^{3x}+y^2e^{3x}+h'(y)=e^{3x}(x^2+y^2)$

$h=C$.

$\therefore$ $\psi=C$

$$yx^2e^{3x}+y^3\frac{e^{3x}}{3}=C$$

1b) $dx + \left(\frac{x}{y} - \sin y\right) dy = 0$,

Ans: $\frac{du}{dy} = \frac{\frac{1}{y}}{1} u = \frac{1}{y} u$

$\therefore u = y$

$y\,dx + (x - y\sin y)\,dy = 0$.

$\exists \psi$ s.t. $\frac{\partial \psi}{\partial x} = y$

$\psi = xy + h(y)$

$\frac{\partial \psi}{\partial y} = x + h'(y) = x - y\sin y$

$h'(y) = -y\sin y$

$h = \int y\, d\cos y$

$= y\cos y - \sin y + C$.

$\therefore \psi = C$

$xy + y\cos y - \sin y = C$

3 Solve $\begin{cases} \dfrac{dy}{dt} = \dfrac{y^{1993}\cos\left(e^{(t^{10}+y^{20})^7}\right)}{1+t^4+y^8} \\ y(0) = 0. \end{cases}$

Ans: since $\dfrac{y^{1993}\cos\left(e^{(t^{10}+y^{20})^7}\right)}{1+t^4+y^8}$ are

Smooth, its derivative must be cts.

by uniquess and existence thm,

$y = 0.$

4 Solve $y'' + 5y' + 6y = 0$

Ans: $\lambda^2 + 5\lambda + 6 = 0$

$\lambda = -2 \text{ or } -3$

$\therefore \quad y = c_1 e^{-2t} + c_2 e^{-3t}$

5 Solve $\begin{cases} 4y'' - 4y' + y = 0 \\ y(0) = 2 \\ y'(0) = \frac{1}{3} \end{cases}$

Ans: $4\lambda^2 - 4\lambda + 1 = 0$

$\lambda = \frac{1}{2}$

$y = (c_1 t + c_2) e^{\frac{t}{2}}$

$y(0) = C_2 = 2$

$y'(t) = \frac{1}{2} e^{\frac{t}{2}} (c_1 t + c_2) + e^{\frac{t}{2}} c_1$

$y'(0) = 1 + C_1 = \frac{1}{3}$

$C_1 = -\frac{2}{3}$

$\therefore \ y = 2 e^{\frac{t}{2}} - \frac{2}{3} t e^{\frac{t}{2}}$

6 Solve $y'' + y' + y = 0$

Ans:

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{3} i}{2}$

$\therefore \ y = e^{-\frac{t}{2}} \left(c_1 \sin \frac{\sqrt{3}}{2} t + c_2 \cos \frac{\sqrt{3}}{2} t \right)$